The application of Bernoulli's law in case of non-stationary flow in the cochlea.

Willem Chr. Heerens May 2011.

In order to deduce that the application of Bernoulli's equation for non-stationary flow in a folded tube like the combination of scala tympani and scala vestibuli is allowed without further constraints we have to observe all the necessary conditions and parameters that are involved.

But let me first remark that under a normal pattern of sound pressure stimuli realistic data for the deflections of both the oval window and the round window together with the perilymph in that folded duct in between have deflections that does not exceed 10 micrometer.

That perilymph duct is entirely filled with perilymph, while perilymph is an incompressible fluid.
Another conditioning parameter that might play a role is the value of the viscosity in perilymph. According to literature the perilymph viscosity is approx. three times that of water.
This means that the perilymph cannot be observed as a viscous fluid and in combination with the characteristic diameter of the perilymph duct of 0.3 mm and for 10 micrometer displacement at a frequency of 20 kHz the Reynolds number can be calculated as 0.06. Which means that in none of the possible acoustical stimulations the flow will be turbulent.

The result of this investigation of conditions is that the existing 'vibration' movement meets all the necessary characteristics of a so-called potential flow, which will be a function of time, because the fluid is non-compressible [comparable with water], free of friction [a low viscosity not much higher than water], free of rotation [the flow behaves as a parallel streaming oriented along the core of the perilymph duct].
Besides that gravity does not play a role here. Only the force evoked by the sound stimulus, which result in a time dependent acceleration.

In fact the entire duct is forming one uniform flow tube. And now we can in principle apply for the Bernoulli equation for incompressible, non-stationary, rotation free streaming.
This equation in its general form is given by:

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}\,|\nabla\Phi|^2 + \frac{p}{\rho} - f = '\text{constant}'$$

In which we have to indicate the different symbols.

Φ is the 'velocity potential'.

Now in this computational model we have to realize that for the gradient of this velocity potential – hence the velocity – counts that there exists only a contribution in the x-direction, which is parallel to the core of the perilymph duct and which is realized by the relation:

$$\Phi = x \times v_x \quad \text{or} \quad v_x = \nabla\Phi = '\text{constant}'.$$

Here 'constant' means independent of place coordinates and only dependent on time.

p is the pressure inside the fluid, which will also be time dependent and to be calculated after the insertion of the relations for the other contributions.

ρ is the density and a material constant approx. equal to that of water, being 1000 kg/m^3.

f will then be the 'force potential', from which as a gradient a conservative external force field will result.
In this case this is defined by the acceleration belonging to the force, with here the value:

$$F = F_x = \nabla f$$

and which has its direction of operation parallel to the x-axis.

And now also counts for that acceleration that it is equal to the acceleration of the fluid in the x-direction. In formula given as:

$$F = F_x = \frac{\partial v_x}{\partial t}$$

And because in this case counts that: $F = F_x = \nabla f$, we can also use the notation:

$$\nabla f = \frac{\partial v_x}{\partial t}$$

And because v_x is only a function of the time t, it can also be written in the integral form as:

$$f = \iiint \nabla f \; dxdydz = \iiint \frac{\partial v_x}{\partial t} dxdydz = \frac{\partial}{\partial t} \iiint v_x dxdydz = \frac{\partial \Phi}{\partial t}$$

All these relations inserted in Bernoulli's equation for incompressible, non-stationary, rotation free flow, expressed by:

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2} \, |\nabla \Phi|^2 + \frac{p}{\rho} - f = '\text{constant}'$$

result in:

For $|\nabla \Phi|^2$ counts that it is equal to:

$$v_x^2 = v(t)^2$$

For f now counts the relation:

$$f = \frac{\partial \Phi}{\partial t}$$

Hence after insertion in the Bernoulli equation results:

$$\frac{1}{2} \, v(t)^2 + \frac{p}{\rho} = '\text{constant}'$$

And now for that time dependent ‘constant’ we can make the following choice:

For a velocity $v(t) = 0$ that ‘constant’ will be independent of time and equal to:

$$\frac{p(0)}{\rho}$$

Finally in case the velocity $v(t) \neq 0$ we can write:

$$\tfrac{1}{2}v(t)^2 + \frac{p}{\rho} = \frac{p(0)}{\rho}$$

We can present the pressure p as the time dependent pressure quantity $p(t)$ and introduce the relation:

$$\Delta p(t) = p(t) - p(0)$$

in which $\Delta p(t)$ equals the change in pressure all over the stream tube as a result of the change in the flow velocity from 0 to the uniform flow velocity $v(t)$.

Finally this results in the relation:

$$\Delta p(t) = -\tfrac{1}{2}\rho v(t)^2$$

And this is exactly the equation that relates the everywhere existing time dependent pressure decrease in front of the basilar membrane $\Delta p(t)$ directly with the square of the velocity in the perilymph duct in the cochlea.

Because that velocity is proportional to the time derivative of the sound pressure stimulus evoked in front of the external parts of the ear, the overall relation between that sound pressure signal and the stimulus all over the membrane is purely quadratic.

And this is completely similar to the statement that the pressure stimulus all over the basilar membrane is proportional to the sound energy signal in front of the eardrum.